TRIGONOMETRIC

1. BASIC TRIGONOMETRIC IDENTITIES :

(a) $\sin^2\theta + \cos^2\theta = 1$; $-1 \le \sin\theta \le 1$; $-1 \le \cos\theta \le 1$ $\forall$ $\theta \in R$

(b) $\sec^2\theta - \tan^2\theta = 1$; $|\sec\theta| \ge 1$ $\forall$ $\theta \in R$

(c) $\text{cosec}^2\theta - \cot^2\theta = 1$; $|\text{cosec}\,\theta| \ge 1$ $\forall$ $\theta \in R$

2. IMPORTANT T′ RATIOS:

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$; where $n \in I$

(b) $\sin\frac{(2n+1)\pi}{2} = (-1)^n$ & $\cos\frac{(2n+1)\pi}{2} = 0$ where $n \in I$

(c) $\sin 15°$ or $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75°$ or $\cos\frac{5\pi}{12}$;

$\cos 15°$ or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75°$ or $\sin\frac{5\pi}{12}$;

$\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75°$; $\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15°$

(d) $\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$; $\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$; $\tan\frac{\pi}{8} = \sqrt{2}-1$; $\tan\frac{3\pi}{8} = \sqrt{2}+1$

(e) $\sin\frac{\pi}{10}$ or $\sin 18° = \frac{\sqrt{5}-1}{4}$ & $\cos 36°$ or $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$

3. TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES :

If θ is any angle, then $-\theta$, $90 \pm \theta$, $180 \pm \theta$, $270 \pm \theta$, $360 \pm \theta$ etc. are called **ALLIED ANGLES**.

(a) $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$

(b) $\sin(90°-\theta) = \cos\theta$; $\cos(90°-\theta) = \sin\theta$

(c) $\sin(90°+\theta) = \cos\theta$; $\cos(90°+\theta) = -\sin\theta$

(d) $\sin(180°-\theta) = \sin\theta$; $\cos(180°-\theta) = -\cos\theta$

(e) $\sin(180°+\theta) = -\sin\theta$; $\cos(180°+\theta) = -\cos\theta$

(f) $\sin(270°-\theta) = -\cos\theta$; $\cos(270°-\theta) = -\sin\theta$

(g) $\sin(270°+\theta) = -\cos\theta$; $\cos(270°+\theta) = \sin\theta$

4. TRIGONOMETRIC FUNCTIONS OF SUM OR DIFFERENCE OF TWO ANGLES :

(a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(c) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B) \cdot \sin(A-B)$

(d) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B) \cdot \cos(A-B)$

(e) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ **(f)** $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

5. FACTORISATION OF THE SUM OR DIFFERENCE OF TWO SINES OR COSINES

(a) $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ **(b)** $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ **(d)** $\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

6. TRANSFORMATION OF PRODUCTS INTO SUM OR DIFFERENCE OF SINES & COSINES :

(a) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ **(b)** $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(c) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$ **(d)** $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

7. MULTIPLE ANGLES AND HALF ANGLES :

(a) $\sin 2A = 2\sin A\cos A$; $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$;

$\cos\theta = \cos^2\dfrac{\theta}{2} - \sin^2\dfrac{\theta}{2} = 2\cos^2\dfrac{\theta}{2} - 1 = 1 - 2\sin^2\dfrac{\theta}{2}$

$2\cos^2 A = 1 + \cos 2A$, $2\sin^2 A = 1 - \cos 2A$; $\tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A}$

$2\cos^2\dfrac{\theta}{2} = 1 + \cos\theta$, $2\sin^2\dfrac{\theta}{2} = 1 - \cos\theta$

(c) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$; $\tan\theta = \dfrac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}$

(d) $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$, $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$ **(e)** $\sin 3A = 3\sin A - 4\sin^3 A$

(f) $\cos 3A = 4\cos^3 A - 3\cos A$ **(g)** $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

8. THREE ANGLES:

(a) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$

Note If : **(i)** $A+B+C = \pi$ then $\tan A + \tan B + \tan C = \tan A\tan B\tan C$

(ii) $A+B+C = \dfrac{\pi}{2}$ then $\tan A\tan B + \tan B\tan C + \tan C\tan A = 1$

(b) If $A + B + C = \pi$ then : **(i)** $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

(ii) $\sin A + \sin B + \sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

9. MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC FUNCTIONS:

(a) Min. value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$ where $\theta \in R$

(b) Max. and Min. value of $a\cos\theta + b\sin\theta$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$

(c) If $f(\theta) = a\cos(\alpha+\theta) + b\cos(\beta+\theta)$ where a, b, α and β are known quantities then

$$-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le f(\theta) \le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$$

(d) If A, B, C are the angles of a triangle then maximum value of $\sin A + \sin B + \sin C$ and $\sin A \sin B \sin C$ occurs when $A = B = C = 60^0$

(e) In case a quadratic in $\sin\theta$ or $\cos\theta$ is given then the maximum or minimum values can be interpreted by making a perfect square.

10 SUM OF SINE OR COSINE OF n ANGLES :

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots + \sin\left(\alpha + \overline{n-1}\,\beta\right) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \sin\left(\alpha + \frac{n-1}{2}\beta\right)$$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots + \cos\left(\alpha + \overline{n-1}\,\beta\right) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cos\left(\alpha + \frac{n-1}{2}\beta\right)$$

TRIGONOMETRIC EQUATION

THINGS TO REMEMBER :

1. If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.

2. If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.

3. If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.

4. If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

5. $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

6. $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$. [Note : α is called the principal angle]

7. **TYPES OF TRIGONOMETRIC EQUATIONS :**

(a) Solutions of equations by factorising. Consider the equation ;
$(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$; $\cot x - \cos x = 1 - \cot x \cos x$

(b) Solutions of equations reducible to quadratic equations. Consider the equation :
$3\cos^2 x - 10\cos x + 3 = 0$ and $2\sin^2 x + \sqrt{3}\sin x + 1 = 0$

(c) Solving equations by introducing an Auxilliary argument. Consider the equation :
$\sin x + \cos x = \sqrt{2}$; $\sqrt{3}\cos x + \sin x = 2$; $\sec x - 1 = (\sqrt{2} - 1)\tan x$

Note : Trigonometric equation of the form $a\sin x + b\cos x = c$ can also be solved by changing sinx and cosx into their corresponding tangent of half the angle.

(e) Solving equations by transforming a product of trigonometric functions into a sum. Consider the equation :

(f) Solving equations by a change of variable :

(i) Equations of the form of $a.\sin x + b.\cos x + d = 0$, where a, b & d are real numbers & $a, b \neq 0$ can be solved by changing $\sin x$ & $\cos x$ into their corresponding tangent of half the angle. Consider the equation $3\cos x + 4\sin x = 5$.

(ii) Many equations can be solved by introducing a new variable . eg. the equation $\sin^4 2x + \cos^4 2x = \sin 2x \cdot \cos 2x$ changes to

$$2(y+1)\left(y-\frac{1}{2}\right) = 0 \quad \text{by substituting , } \sin 2x \cdot \cos 2x = y.$$

(g) Solving equations with the use of the Boundness of the functions $\sin x$ & $\cos x$ or by making two perfect squares. Consider the equations :

$$\sin x\left(\cos\frac{x}{4} - 2\sin x\right) + \left(1+\sin\frac{x}{4} - 2\cos x\right) . \cos x = 0 \;;$$

$$\sin^2 x + 2\tan^2 x + \frac{4}{\sqrt{3}}\tan x - \sin x + \frac{11}{12} = 0$$

IMPORTANT POINTS :

1. Many trigonometrical equations can be solved by different methods. The form of solution obtained in different methods may be different. From these different forms of solutions, the students should not think that the answer obtained by one method are wrong and those obtained by another method are correct. The solutions obtained by different methods may be shown to be equivalent by some supplementary transformations.
To test the equivalence of two solutions obtained from two methods, the simplest way is to put values of n = ...–2, –1, 0, 1, 2, 3....... etc. and then to find the angles in $[0, 2\pi]$. If all the angles in both solutions are same, the solutions are equivalent.

2. While manipulating the trigonometrical equation, avoid the danger of losing roots. Generally, some roots are lost by cancelling a common factor from the two sides of an equation. For Ex., suppose we have the equation $\tan x = 2\sin x$. Here by dividing both sides by $\sin x$, we get $\cos x = 1/2$. This is not equivalent to the original equation. Here the roots obtained by $\sin x = 0$, are lost. Thus in place of dividing an equation by a common factor, the students are advised to take this factor out as a common factor from all terms of the equation.

3. While equating one of the factors to zero, take care of the other factor that it should not become infinite. For Ex., if we have the equation $\sin x = 0$, which can be written as $\cos x \tan x = 0$. Here we cannot put $\cos x = 0$, since for $\cos x = 0$, $\tan x = \sin x/\cos x$ is infinite.

4. **Avoid squaring :** When we square both sides of an equation, some extraneous roots appear. Hence it is necessary to check all the solutions found by substituting them in the given equation and omit the solutions not satisfying the given equation.
For Ex. : Consider the equation,
$\sin\theta + \cos\theta = 1$ .....(1)
Squaring we get
$1 + \sin 2\theta = 1$ or $\sin 2\theta = 0$ .....(2)

i.e. $2\theta = n\pi$ or $\theta = n\pi/2$,

This gives $\theta = 0$, $\frac{\pi}{2}, \pi, \frac{3\pi}{2}, ...$

Verification shows that π and $\frac{3\pi}{2}$ do not satisfy the equation as $\sin\pi + \cos\pi = -1, \neq 1$

and $\sin\frac{3\pi}{2} + \cos\frac{3\pi}{2} = -1, \quad \neq 1.$

The reason for this is simple.

The equation (2) is not equivalent to (1) and (2) contains two equations : $\sin\theta + \cos\theta = 1$ and $\sin\theta + \cos\theta = -1$. Therefore we get extra solutions.

Thus if squaring is must, verify each of the solution.

5. **Some necessary restrictions :**

If the equation involves tanx, secx, take $\cos x \neq 0$. If cot x or cosec x appear, take $\sin x \neq 0$.

If log appear in the equation, i.e. $\log[f(\theta)]$ appear in the equation, use $f(\theta) > 0$ and base of $\log > 0, \neq 1$.

Also note that $\sqrt{[f(\theta)]}$ is always positive, for Ex. $\sqrt{\sin^2\theta} = |\sin\theta|$, not $\pm\sin\theta$.

6. **Verification :** Student are advice to check whether all the roots obtained by them satisfy the equation and lie in the domain of the variable of the given equation.

8. **TRIGONOMETRIC INEQUALITIES :** There is no general rule to solve a Trigonometric inequations and the same rules of algebra are valid except the domain and range of trigonometric functions should be kept in mind.

Consider the Ex.s : $\log_2\left(\sin\frac{x}{2}\right) < -1$; $\sin x\left(\cos x + \frac{1}{2}\right) \leq 0$; $\sqrt{5 - 2\sin 2x} \geq 6\sin x - 1$